Extension of the discrete KP hierarchy

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Abstract. We introduce the discrete hierarchy which naturally generalizes well known discrete KP hierarchy.

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1. Introduction

The main purpose of this paper is to show how the well known discrete KP hierarchy [1] can be generalized by adding additional multi-times. What we get in results is referred to as extended discrete KP while any subsystem of it corresponding to multi-time $t^{(n)} \equiv (t_1^{(n)} \equiv x^{(n)}, t_2^{(n)}, ...)$ we call nth discrete KP hierarchy.

We show in the paper that nth discrete KP in fact is equivalent to bi-infinite sequence of copies of differential KP hierarchy whose Lax operators are connected with each other by compatible 'gauge' transformations. The compatibility of the latter turned out to be equivalent to equations of motion which represent the first flow in nth discrete KP.

The paper is organized as follows. After giving some notations in Section 2, in Section 3 we introduce and discuss the extension of the discrete KP. Section 4 is devoted to provide relationship between nth discrete KP and sequence of differential KP.

2. The differential and discrete KP hierarchy

Let us recall some basic facts about the differential KP hierarchy in the spirit of Sato theory [2], [3], [4]. This approach essentially is based on the calculus of the pseudo-differential operators (ΨDO's) [5]. For reasons of completeness a certain amount of notation has to be introduced.

The unknown functions (fields) depend on spatial variable $t_1 \equiv x \in \mathbf{R}^1$ and some evolution parameters $t_2, t_3, ...$ The symbols ∂ and ∂_p stand for derivation with respect to x and t_p, respectively. In what follows the symbol t denotes KP multi-time, i.e. infinite set of evolution parameters $(t_1, t_2, t_3, ...)$. Let R be a commutative ring consisting of all smooth functions $a = a(x)$. Then noncommutative ring $R[\partial, \partial^{-1})$ of ΨDO's consists of all formal expressions

$$A = \sum_{i=-\infty}^{N} a_i(x)\partial^i, \quad N \in \mathbf{Z}$$

with coefficients in R. One says that ΨDO A is of order N. The operator $\partial : R \to R$ is entirely defined by generalized Leibniz rule

$$\partial^i \circ a = \sum_{j=0}^{\infty} \binom{i}{j} a^{(j)} \partial^{i-j}$$

where $a^{(j)} \equiv \partial^j a$. The adjoint of A is given by

$$A^* = \sum_{i=-\infty}^{N} (-\partial)^i \circ a_i.$$

The important rôle in the theory plays decomposition of elements of $R[\partial, \partial^{-1})$ into

positive (differential) and negative (integral) parts. We denote

$$A_+ = \sum_{i\geq 0} a_i(x)\partial^i, \quad A_- = \sum_{i\leq -1} a_i(x)\partial^i.$$

respectively.

It is convenient to introduce a formal dressing operator $\hat{w} = 1 + \sum_{k\in\mathbf{N}} w_k\partial^{-k}$. Then the KP hierarchy can be represented via Sato–Wilson equations

$$\partial_p\hat{w} = -(\hat{w}\partial^p\hat{w}^{-1})_-\hat{w} = (\hat{w}\partial^p\hat{w}^{-1})_+\hat{w} - \hat{w}\partial^p \tag{1}$$

or equivalently as Lax equations

$$\partial_p\mathcal{Q} = [(\mathcal{Q}^p)_+, \mathcal{Q}] \equiv (\mathcal{Q}^p)_+\mathcal{Q} - \mathcal{Q}(\mathcal{Q}^p)_+. \tag{2}$$

on first-order ΨDO $\mathcal{Q} = \hat{w}\partial\hat{w}^{-1} = \partial + \sum_{k=1}^{\infty} u_k(t)\partial^{-k}$. The very important observation is that evolution equations of the KP hierarchy are solved in terms of single τ-function satisfying an infinite set of bilinear equations which are encoded in the fundamental bilinear identity

$$\mathrm{res}_z[\psi(t,z)\psi^*(t^{'},z)] \equiv \frac{1}{2\pi i}\oint_0 \psi(t,z)\psi^*(t^{'},z)dz = 0. \tag{3}$$

Recall that formal Backer – Akhiezer wave function ψ and its conjugate ψ^* entering fundamental identity are related to KP τ-function via

$$\psi(t,z) = \frac{\tau(t-[z^{-1}])}{\tau(t)}\exp(\xi(t,z)), \quad \psi^*(t,z) = \frac{\tau(t+[z^{-1}])}{\tau(t)}\exp(-\xi(t,z))$$

with $\xi(t,z) = \sum_{p=1}^{\infty} t_p z^p$ and $[z^{-1}] = (1/z, 1/(2z^2), 1/(3z^3), ...)$. Then the bilinear identity (3) becomes

$$\sum_{k=0}^{\infty} p_k(-2a)p_{k+1}(\tilde{D}_t)\tau\circ\tau = 0, \quad \forall a = (a_1, a_2, ...).$$

A few remarks are in order. For given polynomial $p(\partial/\partial t_1, \partial/\partial t_2, ...)$ in $\partial/\partial t_i$, one defines

$$\begin{aligned} &p(D_{t_1}, D_{t_2}, ...)f\circ g \\ &= p\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, ...\right) f(t_1+u_1, t_2+u_2, ...)g(t_1-u_1, t_2-u_2, ...)\bigg|_{u=0}. \end{aligned}$$

In what follows $\tilde{D}_t \equiv (D_{t_1}, \frac{1}{2}D_{t_2}, \frac{1}{3}D_{t_3}, ...)$. It is worth also to recall following identity†.:

$$\frac{1}{k!}\left(\frac{d}{du}\right)^k f(t+[u])g(t-[u])\bigg|_{u=0} = p_k(\tilde{D})f\circ g \tag{4}$$

† here $t = (t_1, t_2, ...)$

which will be useful in the following. Here Schur polynomials $p_k(t)$ are defined through

$$\exp\left(\sum_{p=1}^{\infty} t_p z^p\right) = \sum_{k=0}^{\infty} z^k p_k(t).$$

The discrete KP is a KP hierarchy where continuous space variable gets replaced by a discrete i-variable. More exactly, equations of motion of discrete KP are encoded by Lax equation

$$\frac{\partial Q}{\partial t_p} = [Q_+^p, Q]$$

on difference operator $Q = \Lambda + \sum_{k\in\mathbf{N}} a_{k-1}\Lambda^{1-k}$. The main reference in this context is the paper of Ueno and Takasaki [1]. This hierarchy as well as a large class of its solutions is well described in [6]. What we learn from this work (see also [7]) is that the discrete KP is tantamount to bi-infinite sequence of differential KP copies 'glued' together by Darboux-Bäcklund (DB) transformations. This leads to certain bilinear relations connecting the consecutive KP τ-functions. It should be noted that integrable systems which are chains of infinitely many copies of KP-type differential hierarchies turn out to be useful in matrix models [8], [9], [7]. In the work [10] by Dickey, it was shown that the discrete KP hierarchy is the most natural generalization of the modified KP.

3. Extended discrete KP hierarchy

Throughout the paper, we will be dealing with $\infty\times\infty$ matrices. Given the shift operator $\Lambda = (\delta_{i,j-1})_{i,j\in\mathbf{Z}}$ and 'spectral' parameter z one considers the following spaces of the difference operators†:

$$\mathcal{D}^{(n,r)} = \left\{ \sum_{-\infty\ll k<\infty} l_k z^{k(n-1)}\Lambda^{r-kn} \right\} = \mathcal{D}_-^{(n,r)} + \mathcal{D}_+^{(n,r)}$$

with $l_k \equiv (l_k(i))_{i\in\mathbf{Z}}$ being bi-infinite diagonal matrices. One can easily check, the following properties:

$$\mathcal{D}^{(n,r_1)}\cdot\mathcal{D}^{(n,r_2)} \subset \mathcal{D}^{(n,r_1+r_2)}, \quad \Lambda\cdot\mathcal{D}^{(n,r)} \subset \mathcal{D}^{(n,r+1)},$$

$$\mathcal{D}^{(n,r)}\cdot\Lambda \subset \mathcal{D}^{(n,r+1)}, \quad z^{n-1}\mathcal{D}^{(n,r)} \subset \mathcal{D}^{(n,r+n)}.$$

Remark 1. In the case $n = 1$, a dependence of $\mathcal{D}^{(1,r)}$ on r make no sense because $L \in \mathcal{D}^{(1,r)}$ does not depend on z.

† z acts as component-wise multiplication.

The splitting of $\mathcal{D}^{(n,r)}$ into 'negative' and 'positive' parts is defined as follows:

$$\mathcal{D}_-^{(n,r)} = \left\{ \sum_{r-kn\leq -1} l_k z^{k(n-1)}\Lambda^{r-kn} \right\}, \quad \mathcal{D}_+^{(n,r)} = \left\{ \sum_{r-kn\geq 0} l_k z^{k(n-1)}\Lambda^{r-kn} \right\}.$$

In the following we assume that the entries of l_k's may depend on multi-time $t \equiv (t_p^{(n)})_{p,n\in\mathbf{N}}$. For corresponding time derivatives we use following notation: $\partial_p^{(n)} = \partial/\partial t_p^{(n)}$ and $\partial^{(n)} = \partial/\partial x^{(n)}$, where $x^{(n)} \equiv t_1^{(n)}$.

The phase space $\mathcal{M}$ consists of the entries of diagonal matrices $w_k = (w_k(i))_{i\in\mathbf{Z}},\ k \in \mathbf{N}$. For each $n \in \mathbf{N}$, we define the 'wave' operator

$$S^{(n)} = I + \sum_{k\in\mathbf{N}} w_k z^{k(n-1)}\Lambda^{-kn} \in I + \mathcal{D}_-^{(n,0)} \tag{5}$$

and the corresponding Lax operator

$$Q^{(n)} \equiv S^{(n)}\Lambda S^{(n)-1} = \Lambda + \sum_{k\in\mathbf{N}} a_{k-1}^{(n)} z^{k(n-1)}\Lambda^{1-kn} \in \mathcal{D}^{(n,1)}. \tag{6}$$

It is clear, the coordinates $a_k^{(n)}$ are related with original ones by some polynomial relations. For example, from (6) one can read off the following:

$$\begin{aligned}
&a_0^{(n)}(i) = w_1(i) - w_1(i+1),\\
&a_1^{(n)}(i) = w_2(i) - w_2(i+1) + w_1(i-n+1)(w_1(i+1) - w_1(i)),\\
&a_2^{(n)}(i) = w_3(i) - w_3(i+1) + w_1(i-2n+1)(w_2(i+1) - w_2(i))\\
&+w_2(i-n+1)(w_1(i+1) - w_1(i))\\
&+w_1(i-2n+1)w_1(i-n+1)(w_1(i) - w_1(i+1)).
\end{aligned}$$

Now we are in position to define the flows on $\mathcal{M}$ with respect to parameters $t_p^{(n)}$. It will be managed by the equations of motion on the 'wave' operator †

$$\begin{aligned}
z^{p(n-1)}\frac{\partial S}{\partial t_p^{(n)}} &= Q_+^{pn}S - S\Lambda^{pn}, \ \in \mathcal{D}^{(n,pn)}\\
z^{p(n-1)}\frac{\partial (S^{-1})^T}{\partial t_p^{(n)}} &= (S^{-1})^T\Lambda^{-pn} - (Q_+^{pn})^T(S^{-1})^T.
\end{aligned} \tag{7}$$

We should perhaps note that the first and second equations in (7) in fact are equivalent. Evolutions of S induces evolutions of Q in the form of the Lax equations

$$z^{p(n-1)}\frac{\partial Q}{\partial t_p^{(n)}} = [Q_+^{pn}, Q] \quad \in \mathcal{D}^{n,pn+1}. \tag{8}$$

† henceforth we shall employ the short-hand notation S for $S^{(n)}$ and Q for $Q^{(n)}$ whenever this will not lead to a confusion.

One can easily check that $[Q_+^{pn}, Q] = -[Q_-^{pn}, Q]$ is of the same form as l.h.s. of (8) and therefore (8) and equivalent equations (7) are properly defined.

Obviously, the discrete KP hierarchy can be regarded as a subsystem of (7) with respect to infinite set of parameters $t^{(1)} = (t_1^{(1)}, t_2^{(1)}, ...)$. For this reason, we will refer to (7) as extended discrete KP hierarchy. The subsystem of (7) corresponding to evolution parameters $t^{(n)} = (t_1^{(n)}, t_2^{(n)}, ...)$ we will call, according to [11], nth discrete KP hierarchy.

In the following it will be useful also to consider evolution equations

$$z^{p(n-1)} \frac{\partial Q^r}{\partial t_p^{(n)}} = [Q_+^{pn}, Q^r], \quad r \in \mathbf{Z} \tag{9}$$

being consequences of (8). It is easy to see that rth power of Q is of the form

$$Q^r = \Lambda^r + \sum_{k\in\mathbf{N}} a_{k-1}^{(n,r)} z^{k(n-1)} \Lambda^{r-kn} \in \mathcal{D}^{(n,r)}.$$

with diagonal matrices $a_k^{(n,r)}$ whose entries are polynomially expressed via original coordinates $w_k(i)$.

4. nth discrete KP

Define $\chi(z) = (z^i)_{i\in\mathbf{Z}}$, $\chi^*(z) = \chi(z^{-1})$ and wave vectors

$$\Psi(t,z) = W\chi(z), \quad \Psi^*(t,z) = (W^{-1})^T \chi^*(z) \tag{10}$$

where $W \equiv S \exp(\sum_{p=1}^{\infty} t_p^{(n)} \Lambda^p)$. Discrete linear system

$$\begin{gathered} Q\Psi(t,z) = z\Psi(t,z), \quad Q^T\Psi^*(t,z) = z\Psi^*(t,z), \\ z^{p(n-1)}\partial_p^{(n)}\Psi = Q_+^{pn}\Psi, \quad z^{p(n-1)}\partial_p^{(n)}\Psi^* = -(Q_+^{pn})^T\Psi^* \end{gathered} \tag{11}$$

are evident consequence of (7) and (10). ¿From (10), it follows

$$\begin{aligned} \Psi_i(t,z) &= z^i(1 + w_1(i)z^{-1} + w_2(i)z^{-2} + ...)e^{\xi(t^{(n)},z)} \\ &= z^i(1 + w_1(i)\partial^{(n)-1} + w_2(i)\partial^{(n)-2} + ...)e^{\xi(t^{(n)},z)} \\ &\equiv z^i \hat{w}_i(\partial^{(n)})e^{\xi(t^{(n)},z)} \equiv z^i\psi_i(t,z). \end{aligned}$$

Next, we are going to show equivalence of nth discrete KP to bi-infinite sequence of differential KP copies "glued" together by compatible gauge transformations one of which can be recognized as DB transformation mapping $\mathcal{Q}_i \equiv \hat{w}_i\partial^{(n)}\hat{w}_i^{-1}$ to $\mathcal{Q}_{i+n} \equiv \hat{w}_{i+n}\partial^{(n)}\hat{w}_{i+n}^{-1}$.

Proposition 1. *The following three statements are equivalent:*

(i) The wave vector $\Psi(t,z)$ satisfies discrete linear system

$$Q^r\Psi(t,z) = z^r\Psi(t,z), \quad z^{n-1}\partial^{(n)}\Psi = Q_+^n\Psi, \quad r \in \mathbf{Z}; \tag{12}$$

(ii) The components ψ_i *of the vector* $\psi \equiv (\psi_i = z^{-i}\Psi_i)_{i\in\mathbf{Z}}$ *satisfy*

$$G_i^{(r)}\psi_i(t,z) = z\psi_{i+n-r}(t,z), \quad H_i\psi_i(t,z) = z\psi_{i+n}(t,z) \tag{13}$$

with $H_i \equiv \partial^{(n)} - \sum_{s=1}^n a_0^{(n)}(i+s-1)$ *and*

$$G_i^{(r)} \equiv H_i + a_0^{(n,r)}(i+n-r)$$

$$+a_1^{(n,r)}(i+n-r)H_{i-n}^{-1} + a_2^{(n,r)}(i+n-r)H_{i-2n}^{-1}H_{i-n}^{-1} + ...;$$

(iii) For sequence of $\partial^{(n)}$*-dressing operators* $\{\hat{w}_i,\ i\in\mathbf{Z}\}$ *the equations*

$$G_i^{(r)}\hat{w}_i = \hat{w}_{i+n-r}\partial^{(n)}, \quad H_i\hat{w}_i = \hat{w}_{i+n}\partial^{(n)} \tag{14}$$

hold.

Remark 2. Since $Q^0 = I$ and $a_k^{(n,0)}(i) = 0$, we have in this case $G_i^{(0)} = H_i$.

Proof of Proposition 1. Rewrite equations (12) in explicit form

$$\Psi_{i+r} + a_0^{(n,r)}(i)z^{n-1}\Psi_{i+r-n} + a_1^{(n,r)}(i)z^{2(n-1)}\Psi_{i+r-2n} + ... = z^r\Psi_i,$$

$$z^{n-1}\partial^{(n)}\Psi_i = \Psi_{i+n} + \left(\sum_{s=1}^n a_0^{(n)}(i+s-1)\right)\Psi_i.$$

In terms of wave functions ψ_i the latter is rewritten as

$$z\psi_{i+r} + a_0^{(n,r)}(i)\psi_{i+r-n} + \frac{1}{z}a_1^{(n,r)}(i)\psi_{i+r-2n} + ... = z\psi_i, \tag{15}$$

$$\partial^{(n)}\psi_i = z\psi_{i+n} + \left(\sum_{s=1}^n a_0^{(n)}(i+s-1)\right)\psi_i. \tag{16}$$

One sees that equation (16) coincides with second one in (14). Shifting $i \to i-r+n$ in (15) and combining it with (16) one can obtain first equation in (14). Therefore we proved (i) $\Rightarrow$ (ii). The converse also easily can be showed by returning to the functions Ψ_i. The equivalence (ii) $\Leftrightarrow$ (iii) follows from representation $\psi_i(t,z) = \hat{w}_i e^{\xi(t^{(n)},z)}$. $\square$

Let us write down in explicit form equations of motion coded in Lax equation

$$z^{n-1}\partial^{(n)}Q^r = [Q_+^n, Q^r] \tag{17}$$

being consistency condition of linear discrete system (12). We have

$$\partial^{(n)}a_k^{(n,r)}(i) = a_{k+1}^{(n,r)}(i+n) - a_{k+1}^{(n,r)}(i)$$
$$+a_k^{(n,r)}(i)\left(\sum_{s=1}^n a_0^{(n)}(i+s-1) - \sum_{s=1}^n a_0^{(n)}(i+s+r-(k+1)n-1)\right), \tag{18}$$
$$k = 0, 1, ...$$

where $a_0^{(n,r)}(i) = \sum_{s=1}^{r} a_0^{(n)}(i+s-1)$ in the case $r \in \mathbf{N}$ and $a_0^{(n,r)}(i) = -\sum_{s=1}^{-r} a_0^{(n)}(i-s)$ when $r \in -\mathbf{N}$. Notice that simplest form of these flows is in original coordinates:

$$\partial^{(n)} w_k(i) = w_{k+1}(i+n) - w_{k+1}(i) + w_k(i)(w_1(i) - w_1(i+n)).$$

The system (18) allows for obvious reductions specified by conditions $a_k^{(n,r)}(i) \equiv 0$ when $k \geq k_0$ with $k_0 \in \mathbf{N}$. Let us spend a few lines to list a collection of integrable differential-difference systems known from literature which can be derived as reductions of the general system (18).

Remark 3. In what follows, when we say that given system coincides with that in some reference it means that these systems are the same up to very simple — 'cosmetic' — transformations.

Remark 4. Most part of examples below can be found in [17]. Unfortunately the case $r \leq -1$ in this work was overlooked (see example 4).

Example 1. Consider the case $n = 1, r = 2, k_0 = 2$. Corresponding reduction of (18) reads†

$$a_0'(i) + a_0'(i+1)$$

$$= (a_0(i) + a_0(i+1))(a_0(i) - a_0(i+1)) + a_1(i+1) - a_1(i),$$

$$a_1'(i) = 0.$$

Actually, we have in this case one-field lattice‡

$$r_i' + r_{i+1}' = r_i^2 - r_{i+1}^2 + \nu_i, \tag{19}$$

with $\nu_i = a_1(i+1) - a_1(i)$ being some constants. As is known the lattice (19) describes elementary Darboux transformation for Schrödinger operator $L = \partial^2 + q(x)$. An interesting property of the lattice (19) is that it reduces to Painlevé transcedents P_4 and P_5 due to imposing periodicity conditions $r_{i+N} = r_i, \quad \nu_{i+N} = \nu_i$ for $N = 3$ and $N = 4$, respectively [13].

Example 2. In the case $n = 1,\ r = 1, k_0 \geq 2$ we obtain well known generalized Toda systems known also as Kupershmidt ones [12]

$$a_0'(i) = a_1(i+1) - a_1(i),$$

$$a_k'(i) = a_k(i)\,(a_0(i) - a_0(i-k)) \tag{20}$$

$$+ a_{k+1}(i+1) - a_{k+1}(i), \quad k = 1, ..., k_0 - 1.$$

† here and in what follows $\ ' \equiv \partial/\partial x^{(n)}$ with corresponding n.

‡ For one-field lattices we use notation $a_0(i) = r_i$

In particular if $k_0 = 2$ we obtain ordinary Toda lattice in polynomial form

$$
\begin{aligned}
a_0'(i) &= a_1(i+1) - a_1(i), \\
a_1'(i) &= a_1(i)\left(a_0(i) - a_0(i-1)\right).
\end{aligned} \tag{21}
$$

Defining u_i by relation $a_0(i) = -u_i'$ and $a_1(i) = \exp(u_{i-1} - u_i)$ we arrive at more familiar exponential form of the Toda lattice $u_i'' = e^{u_{i-1}-u_i} - e^{u_i - u_{i+1}}$.

Example 3. Let $n \geq 2,\ r = 1,\ k_0 = 1$. This choice corresponds to Bogoyavlenskii lattices [14]

$$
r_i' = r_i\left(\sum_{s=1}^{n-1} r_{i+s} - \sum_{s=1}^{n-1} r_{i-s}\right). \tag{22}
$$

Example 4. In the case $n = 1,\ r = -1,\ k_0 = 2$ we have Belov–Chaltikian lattice [16]

$$
\begin{aligned}
a_0'(i) &= a_1(i+1) - a_1(i+2) + a_0(i)\left(a_0(i+1) - a_0(i-1)\right), \\
a_1'(i) &= a_1(i)\left(a_0(i) - a_0(i-3)\right).
\end{aligned}
$$

Example 5. In the case $n = 1,\ r = 2,\ k_0 = 3$ we have the system

$$
\begin{aligned}
a_0'(i) + a_0'(i+1) &= a_0^2(i) - a_0^2(i+1) + a_1(i+1) - a_1(i), \\
a_1'(i) &= a_2(i+1) - a_2(i), \\
a_2'(i) &= a_2(i)(a_0(i) - a_0(i-1)).
\end{aligned} \tag{23}
$$

As can be checked the Miura-like transformation

$$
a_0(i) = b(i+1) - b(i),\ a_1(i) = a(i),\ a_2(i) = \frac{c(i)}{c(i-1)}
$$

define a mapping of solutions of the system

$$
\begin{aligned}
a'(i) &= \frac{c(i+1)}{c(i)} - \frac{c(i)}{c(i-1)}, \\
b'(i) + b'(i+1) &= a(i) - (b(i+1) - b(i))^2, \\
c'(i) &= c(i)(b(i+1) - b(i))
\end{aligned}
$$

which appears in [19] into solutions of the lattice (23). As was observed in [17], higher counterpart of (23) is the Blaszak–Marciniak lattice [15].

Define an infinite set of ΨDO's $\{G_i^{(\ell,r)},\ i, \ell, r \in \mathbf{Z}\}$ by means of the following recurrence relations:

$$
\begin{aligned}
G_i^{(\ell+1,r)} &= G_{i+n}^{(\ell,r)} H_i, \quad \ell = 0, 1, 2, ... \\
G_i^{(\ell-1,r)} &= G_{i-n}^{(\ell,r)} H_{i-n}^{-1}, \quad \ell = 0, -1, -2, ...
\end{aligned} \tag{24}
$$

with

$$G_i^{(0,r)} \equiv G_{i-n}^{(r)} H_{i-n}^{-1} = 1 + a_0^{(n,r)}(i-r) H_{i-n}^{-1}$$

$$+ a_1^{(n,r)}(i-r) H_{i-2n}^{-1} H_{i-n}^{-1} + a_2^{(n,r)}(i-r) H_{i-3n}^{-1} H_{i-2n}^{-1} H_{i-n}^{-1} + ...$$

It is important to observe that

$$a_\ell^{(n,r)}(i + \ell n - r) = \mathrm{res}_{\partial^{(n)}} G_i^{(\ell,r)}. \tag{25}$$

Proposition 2. *Following auxiliary equations hold:*

$$G_i^{(\ell,r)} \psi_i = z^\ell \psi_{i+\ell n-r}. \tag{26}$$

Proof. By induction. Let $\ell = 0$, then

$$G_i^{(0,r)} \psi_i = G_{i-n}^{(r)} H_{i-n}^{-1} \psi_i = z^{-1} G_{i-n}^{(r)} \psi_{i-n} = \psi_{i-r}.$$

Now suppose that (26) is true for some l, then

$$G_i^{(\ell+1,r)} \psi_i = G_{i+n}^{(\ell,r)} H_i \psi_i = z G_{i+n}^{(\ell,r)} \psi_{i+n} = z^{\ell+1} \psi_{i+(\ell+1)n-r}.$$

This proves (26) for positive integers ℓ. The similar arguments are used for negative ℓ's. □

As consequence of the proposition, we obtain $G_i^{(\ell,\ell n)} = \mathcal{Q}_i^\ell$. Notice that the equation (26) in equivalent form is rewritten as

$$G_i^{(\ell,r)} \hat{w}_i = \hat{w}_{i+\ell n-r} \partial^{(n)\ell}. \tag{27}$$

Proposition 3. *The relation*

$$G_{i+\ell_2 n-r_2}^{(\ell_1,r_1)} G_i^{(\ell_2,r_2)} = G_i^{(\ell_1+\ell_2,r_1+r_2)} \tag{28}$$

holds.

Proof. Taking into account (27), we obtain that left multiplication of l.h.s. and r.h.s. of (28) on $\hat{w}_i$ gives the same, namely $\hat{w}_{i+(\ell_1+\ell_2)n-r_1-r_2}(\partial^{(n)})^{\ell_1+\ell_2}$. This proves proposition. □

As we have mentioned above, consistency condition of linear system (12) being expressed in the form of Lax equation reads in explicit form as lattice (18). As consequence of proposition 3 we obtain that this system guarantee the validity of permutation relations

$$G_{i+\ell_2 n-r_2}^{(\ell_1,r_1)} G_i^{(\ell_2,r_2)} = G_{i+\ell_4 n-r_4}^{(\ell_3,r_3)} G_i^{(\ell_4,r_4)} \tag{29}$$

with arbitrary integers $\{\ell_k, r_k\}_{k=1}^4$ such that $\ell_1 + \ell_2 = \ell_3 + \ell_4$ and $r_1 + r_2 = r_3 + r_4$. It is clear that permutation relation (29) can be extended on that with arbitrary number of cofactors. In addition since $G_i^{(0,0)} = 1$ we have

$$G_i^{(\ell,r)^{-1}} = G_{i+\ell n-r}^{(-\ell,-r)}$$

¿From the above we learn that the system (18) guarantee that the set of bi-infinite sequences of ΨDO's $\{G_i^{(\ell,r)},\ i\in\mathbf{Z}\}$ endowed with the multiplication rule (28) bears the structure of the group isomorphic to $\mathbf{Z}\times\mathbf{Z}$.

Proposition 4. *By virtue of (27) and its consistency condition (29), $\partial^{(n)}$-Lax operators $\mathcal{Q}_i$ are connected with each other by invertible compatible gauge transformations*

$$\mathcal{Q}_{i+\ell n-r} = G_i^{(\ell,r)}\mathcal{Q}_i G_i^{(\ell,r)^{-1}}. \tag{30}$$

Remark 5. Since $\mathcal{Q}_i^\ell = G_i^{(\ell,\ell n)}$, the relation (30) in the case $r=\ell n$ becomes trivial identity.

Proof of proposition 4. Taking into account (27), we have

$$\mathcal{Q}_{i+\ell n-r} = \hat{w}_{i+\ell n-r}\partial\hat{w}^{-1}_{i+\ell n-r} = (G_i^{(\ell,r)}\hat{w}_i\partial^{(n)-1})\partial^{(n)}(\partial^{(n)}\hat{w}_i^{-1}G_i^{(\ell,r)-1})$$

$$= G_i^{(\ell,r)}\hat{w}_i\partial^{(n)}\hat{w}_i^{-1}G_i^{(\ell,r)-1} = G_i^{(\ell,r)}\mathcal{Q}_iG_i^{(\ell,r)-1}.$$

The mapping $\mathcal{Q}_i\to\tilde{\mathcal{Q}}_i=\mathcal{Q}_{i+m}$, where $m=\ell n-r$ we denote as s_m.

Let $m_1=\ell_1 n-r_1$ and $m_2=\ell_2 n-r_2$. By virtue of (29), where $\ell_3=\ell_2$, $\ell_4=\ell_1$, $r_3=r_2$ and $r_4=r_1$ we get

$$\mathcal{Q}_{i+m_1+m_2} = G^{(\ell_1,r_1)}_{i+m_2}\mathcal{Q}_{i+m_2}G^{(\ell_1,r_1)-1}_{i+m_2}$$

$$= G^{(\ell_1,r_1)}_{i+m_2}G_i^{(r_2)}\mathcal{Q}_iG_i^{(\ell_2,r_2)-1}G^{(\ell_1,r_1)-1}_{i+m_2} = G^{(\ell_2,r_2)}_{i+m_1}G_i^{(\ell_1,r_1)}\mathcal{Q}_iG_i^{(\ell,r_1)-1}G^{(\ell_2,r_2)-1}_{i+m_1}$$

$$= G^{(\ell_2,r_2)}_{i+m_1}\mathcal{Q}_{i+m_1}G^{(\ell_2,r_2)-1}_{i+m_1}.$$

¿From this it follows pairwise compatibility of transformations s_{m_1} and s_{m_2} for any integers m_1 and m_2. So we can write $s_{m_1}\circ s_{m_2}=s_{m_2}\circ s_{m_1}$. The inverse maps s_m^{-1} are well-defined by the formula $\mathcal{Q}_{i-\ell n+r}=G^{(\ell,r)-1}_{i-\ell n+r}\mathcal{Q}_iG^{(\ell,r)}_{i-\ell n+r}=G_i^{(-\ell,-r)}\mathcal{Q}_iG_i^{(-\ell,-r)-1}$. □

Rewrite second equation in (13) as $z^{n-1}H_i\Psi_i(t,z)=\Psi_{i+n}(t,z)=(\Lambda^n\Psi)_i$. ¿From this we derive

$$z^{k(1-n)}(\Lambda^{kn}\Psi)_i = H_{i+(k-1)n}\ldots H_{i+n}H_i\Psi_i,$$

$$z^{k(n-1)}(\Lambda^{-kn}\Psi)_i = H^{-1}_{i-kn}\ldots H^{-1}_{i-2n}H^{-1}_{i-n}\Psi_i.$$

These relations make one-to-one connection between difference operators

$$P=\sum_{k\in\mathbf{Z}} z^{k(1-n)}p_k(t)\Lambda^{kn}\in\mathcal{D}^{(n,0)}$$

and sequences of $\partial^{(n)}$-pseudo-differential operators $\{\mathcal{P}_i,\ i\in\mathbf{Z}\}$ mapping the upper triangular part of given matrix (including main diagonal) into the differential parts of

$\mathcal{P}_i$'s and the lower triangular part of the matrix to the purely pseudo-differential parts. More exactly, we have $(P\Psi)_i = \mathcal{P}_i\Psi_i$, $(P_-\Psi)_i = (\mathcal{P}_i)_-\Psi_i$ and $(P_+\Psi)_i = (\mathcal{P}_i)_+\Psi_i$, where

$$\mathcal{P}_i = \sum_{k>0} p_{-k}(i,t)H^{-1}_{i-kn}...H^{-1}_{i-2n}H^{-1}_{i-n} + \sum_{k\geq 0} p_k(i,t)H_{i+(k-1)n}...H_{i+n}H_i$$

$$= (\mathcal{P}_i)_- + (\mathcal{P}_i)_+.$$

In what follows, we denote $\sigma : P \in \mathcal{D}^{(n,0)} \to \{\mathcal{P}_i,\ i \in \mathbf{Z}\}$. It is easy to check that

$$\sigma : z^{\ell(1-n)}\Lambda^{\ell n-r}Q^r \leftrightarrow \{G_i^{(\ell,r)},\ i \in \mathbf{Z}\}. \tag{31}$$

Proposition 5. *Equations* $z^{p(n-1)}\partial_p^{(n)}\Psi = Q_+^{pn}\Psi,\ p = 2,3,...$ *are equivalent to* $\partial_p^{(n)}\psi_i = (\mathcal{Q}_i^p)_+\psi_i,\ p = 2,3,....$

Proof. Setting $r = \ell n$ in (31) gives

$$\sigma : z^{p(1-n)}Q^{pn} \leftrightarrow \{\mathcal{Q}_i^p, i \in \mathbf{Z}\}.$$

Taking into account this, we have

$$z^i\partial_p^{(n)}\psi_i = \partial_p^{(n)}\Psi_i = z^{p(1-n)}(Q_+^{pn}\Psi)_i = (\mathcal{Q}_i^p)_+\Psi_i = z^i(\mathcal{Q}_i^p)_+\psi_i.$$

The latter proves proposition. $\Box$

We learn from this proposition that nth discrete KP in fact is equivalent to bi-infinite sequence of differential KP hierarchies whose evolution equations can be written as Sato – Wilson equations

$$\partial_p^{(n)}\hat{w}_i = (\mathcal{Q}_i^p)_+\hat{w}_i - \hat{w}_i\partial^{(n)p} \tag{32}$$

where $\hat{w}_i$'s are connected by relations (14) or equivalently as Lax equations

$$\partial_p^{(n)}\mathcal{Q}_i = [(\mathcal{Q}_i^p)_+, \mathcal{Q}_i] \tag{33}$$

where $\mathcal{Q}_i$'s are connected by the gauge transformations (30).

Let us establish equations managing $G_i^{(r)}$-evolutions with respect to KP flows. Differentiating l.h.s. and r.h.s. of (14), by virtue (32), formally leads to evolution equations

$$\partial_p^{(n)}G_i^{(r)} = (\mathcal{Q}_{i+n-r}^p)_+G_i^{(r)} - G_i^{(r)}(\mathcal{Q}_i^p)_+. \tag{34}$$

Notice that in the case $r = n$, the latter becomes Lax equations (33). Using standard arguments, one can show that equations (34) are properly defined individually. Indeed, taking into account (30), one can write $\mathcal{Q}_{i+n-r}^p = G_i^{(r)}\mathcal{Q}_i^pG_i^{(r)-1}$ or $\mathcal{Q}_{i+n-r}^pG_i^{(r)} = G_i^{(r)}\mathcal{Q}_i^p$ for any $p \in \mathbf{N}$. It follows from this that

$$(\mathcal{Q}_{i+n-r}^p)_+G_i^{(r)} - G_i^{(r)}(\mathcal{Q}_i^p)_+ = G_i^{(r)}(\mathcal{Q}_i^p)_- - (\mathcal{Q}_{i+n-r}^p)_-G_i^{(r)}.$$

Thus r.h.s. of (34) as well as l.h.s. is a ΨDO of zero order. Moreover, in the case $r = 0$, i.e. when $G_i^{(0)} = H_i$, r.h.s. of (34) is zeroth order differential operator or simply

function. It is easy now to establish $G_i^{(\ell,r)}$-evolutions with respect to KP flows. This states following proposition.

Proposition 6. *By virtue of (24) and (34), we have*

$$\partial_p^{(n)} G_i^{(\ell,r)} = (\mathcal{Q}^p_{i+\ell n-r})_+ G_i^{(\ell,r)} - G_i^{(\ell,r)} (\mathcal{Q}^p_i)_+. \tag{35}$$

Proof. In the case $\ell = 0$, we obtain

$$\partial_p^{(n)} G_i^{(0,r)} = \partial_p^{(n)} (G^{(r)}_{i-n} H^{-1}_{i-n}) = \{(\mathcal{Q}^p_{i-r})_+ G^{(r)}_{i-n} - G^{(r)}_{i-n} (\mathcal{Q}^p_{i-n})_+\} H^{-1}_{i-n}$$

$$-G^{(r)}_{i-n} H^{-1}_{i-n} \{(\mathcal{Q}^p_i)_+ H_{i-n} - H_{i-n} (\mathcal{Q}^p_{i-n})_+\} H^{-1}_{i-n}$$

$$= (\mathcal{Q}^p_{i-r})_+ G_i^{(0,r)} - G_i^{(0,r)} (\mathcal{Q}^p_i)_+.$$

Since $G_i^{(1,r)} = G_i^{(r)}$, equation (35) in the case $\ell = 1$ immediately follows from (34). The proof of (35) proceeds by induction. Assume that (35) is valid for some ℓ, then

$$\partial_p^{(n)} G_i^{(\ell+1,r)} = \partial_p^{(n)} (G^{(\ell,r)}_{i+n} H_i) = \{(\mathcal{Q}^p_{i+(\ell+1)n-r})_+ G^{(\ell,r)}_{i+n} - G^{(\ell,r)}_{i+n} (\mathcal{Q}^p_{i+n})_+\} H_i$$

$$+G^{(\ell,r)}_{i+n} \{(\mathcal{Q}^p_{i+n})_+ H_i - H_i (\mathcal{Q}^p_i)_+\}$$

$$= (\mathcal{Q}^p_{i+(\ell+1)n-r})_+ G_i^{(\ell+1,r)} - G_i^{(\ell+1,r)} (\mathcal{Q}^p_i)_+.$$

This proves (35) for positive integers ℓ. By similar arguments equation (35) is showed for negative ℓ. □

Proposition 7. *Equations (35) are pairwise compatible.*

Proof. One must to show that permutation relation (29) is invariant with respect to KP flows. With the identities $\ell_1 + \ell_2 = \ell_3 + \ell_4$ and $r_1 + r_2 = r_3 + r_4$, we have

$$\partial_p^{(n)} (G^{(\ell_1,r_1)}_{i+\ell_2 n-r_2} G_i^{(\ell_2,r_2)})$$

$$= \{(\mathcal{Q}^p_{i+(\ell_1+\ell_2)n-r_1-r_2})_+ G^{(\ell_1,r_1)}_{i+\ell_2 n-r_2} - G^{(\ell_1,r_1)}_{i+\ell_2 n-r_2} (\mathcal{Q}^p_{i+\ell_2 n-r_2})_+\} G_i^{(\ell_2,r_2)}$$

$$+G^{(\ell_1,r_1)}_{i+\ell_2 n-r_2} \{(\mathcal{Q}^p_{i+\ell_2 n-r_2})_+ G_i^{(\ell_2,r_2)} - G_i^{(\ell_2,r_2)} (\mathcal{Q}^p_i)_+\}$$

$$= (\mathcal{Q}^p_{i+(\ell_1+\ell_2)n-r_1-r_2})_+ G^{(\ell_1,r_1)}_{i+\ell_2 n-r_2} G_i^{(\ell_2,r_2)} - G^{(\ell_1,r_1)}_{i+\ell_2 n-r_2} G_i^{(\ell_2,r_2)} (\mathcal{Q}^p_i)_+$$

$$= (\mathcal{Q}^p_{i+(\ell_3+\ell_4)n-r_3-r_4})_+ G^{(\ell_4,r_4)}_{i+\ell_3 n-r_3} G_i^{(\ell_3,r_3)} - G^{(\ell_4,r_4)}_{i+\ell_3 n-r_3} G_i^{(\ell_3,r_3)} (\mathcal{Q}^p_i)_+$$

$$= \{(\mathcal{Q}^p_{i+(\ell_3+\ell_4)n-r_3-r_4})_+ G^{(\ell_4,r_4)}_{i+\ell_3 n-r_3} - G^{(\ell_4,r_4)}_{i+\ell_3 n-r_3} (\mathcal{Q}^p_{i+\ell_3 n-r_3})_+\} G_i^{(\ell_3,r_3)}$$

$$+G^{(\ell_4,r_4)}_{i+\ell_3 n-r_3} \{(\mathcal{Q}^p_{i+\ell_3 n-r_3})_+ G_i^{(\ell_3,r_3)} - G_i^{(\ell_3,r_3)} (\mathcal{Q}^p_i)_+\} = \partial_p^{(n)} (G^{(\ell_4,r_4)}_{i+\ell_3 n-r_3} G_i^{(\ell_3,r_3)}).$$

Therefore we proved that equations (35) are pairwise consistent. □

The fact that $\psi_i(t,z) = \hat{w}_i e^{\xi(t^{(n)},z)}$ are KP wave eigenfunctions force they to be expressible via τ-functions

$$\psi_i(t,z) = \frac{\tau_i^{(n)}(t^{(1)}, ..., t^{(n)} - [z^{-1}], ...)}{\tau_i^{(n)}(t^{(1)}, ..., t^{(n)}, ...)} e^{\xi(t^{(n)},z)}$$

where $[z^{-1}] \equiv (1/z, 1/(2z^2), ...)$. Define $\Phi_i^{(n)} = \Phi_i^{(n)}(t)$ via $H_i \Phi_i^{(n)} = 0$ or equivalently through following relation:

$$\partial^{(n)} \Phi_i^{(n)} = \Phi_i^{(n)} \sum_{s=1}^{n} a_0^{(n)}(i+s-1).$$

Taking into consideration the second equation in (34), we get

$$\partial_p^{(n)}(H_i \Phi_i^{(n)}) = (\mathcal{Q}_{i+n}^p)_+ H_i \Phi_i^{(n)} - H_i (\mathcal{Q}_i^p)_+ \Phi_i^{(n)} + H_i \partial_p^{(n)} \Phi_i^{(n)} = 0.$$

¿From this we derive $\partial_p^{(n)} \Phi_i^{(n)} = (\mathcal{Q}_i^p)_+ \Phi_i^{(n)} + \alpha_i \Phi_i^{(n)}$ where α_i's are some constants. Commutativity condition $\partial_p^{(n)} \partial_q^{(n)} \Phi_i^{(n)} = \partial_q^{(n)} \partial_p^{(n)} \Phi_i^{(n)}$ leads to evolution equations for KP eigenfunctions $\partial_p^{(n)} \Phi_i^{(n)} = (\mathcal{Q}_i^p)_+ \Phi_i^{(n)}$, i.e. $\alpha_i = 0$. Thus the relations $\mathcal{Q}_{i+n} = H_i \mathcal{Q}_i H_i^{-1}$ defines DB transformations with eigenfunctions $\Phi_i^{(n)} = \tau_{i+n}^{(n)}/\tau_i^{(n)}$ [8]. It should perhaps to recall that an eigenfunction of Lax operator $\mathcal{Q}$ contains information about DB transformation $\tau \to \overline{\tau} = \Phi\tau$ while the identity

$$\{\tau(t - [z^{-1}]), \overline{\tau}(t)\} + z(\tau(t - [z^{-1}])\overline{\tau}(t) - \overline{\tau}(t - [z^{-1}])\tau(t)) = 0$$

with $\{f, g\} \equiv \partial f \cdot g - \partial g \cdot f$ holds.

Proposition 8. *We have*

$$a_\ell^{(n,r)}(i) = \frac{p_{\ell+1}(\tilde{D}_{t^{(n)}})\tau_{i-\ell n+r}^{(n)} \circ \tau_i^{(n)}}{\tau_{i-\ell n+r}^{(n)} \tau_i^{(n)}}. \tag{36}$$

Proof. To show (36), one need in the well known identity[5]

$$\mathrm{res}_z[(Pe^{xz}) \cdot (Qe^{-xz})] = \mathrm{res}_\partial PQ^* \tag{37}$$

where $P = \sum_{k \in \mathbf{Z}} p_k(x)\partial^k$ and $Q = \sum_{k \in \mathbf{Z}} q_k(x)\partial^k$ are two arbitrary ΨDO's and Q^* is the formal adjoint to Q.

To use further the identity (37) we set $P = G_i^{(\ell,r)} \hat{w}_i$ and $Q = \hat{w}_i^{*-1}$. Taking into account (25), (37) and applying Proposition 2, we get

$$a_\ell^{(n,r)}(i + \ell n - r) = \mathrm{res}_z[(G_i^{(\ell,r)} \hat{w}_i e^{x^{(n)}z})(\hat{w}_i^{*-1} e^{-x^{(n)}z})]$$

$$= \mathrm{res}_z[(G_i^{(\ell,r)} \hat{w}_i e^{\xi(t^{(n)},z)})(\hat{w}_i^{*-1} e^{-\xi(t^{(n)},z)})]$$

$$= \mathrm{res}_z[(G_i^{(\ell,r)} \psi_i(t,z))\psi_i^*(t,z)] = \mathrm{res}_z[z^\ell \psi_{i+\ell n-r}(t,z)\psi_i^*(t,z)]$$

$$= \mathrm{res}_z \left[z^\ell \frac{\tau_{i+\ell n-r}^{(n)}(t^{(1)}, ..., t^{(n)} - [z^{-1}], ...)\tau_i^{(n)}(t^{(1)}, ..., t^{(n)} + [z^{-1}], ...)}{\tau_{i+\ell n-r}^{(n)}(t^{(1)}, ..., t^{(n)}, ...)\tau_i^{(n)}(t^{(1)}, ..., t^{(n)}, ...)} \right]$$

$$= \frac{1}{(\ell+1)!}\left(\frac{d}{du}\right)^{\ell+1}\left[\frac{\tau^{(n)}_{i+\ell n-r}(t^{(1)},...,t^{(n)}-[u],...)\tau^{(n)}_i(t^{(1)},...,t^{(n)}+[u],...)}{\tau^{(n)}_{i+\ell n-r}(t^{(1)},...,t^{(n)},...)\tau^{(n)}_i(t^{(1)},...,t^{(n)},...)}\right]\Bigg|_{u=0}$$

Now using technical identity (4) we obtain

$$a^{(n,r)}_\ell(i+\ell n-r) = \frac{p_{\ell+1}(\tilde{D}_{t^{(n)}})\tau^{(n)}_i \circ \tau^{(n)}_{i+\ell n-r}}{\tau^{(n)}_i\tau^{(n)}_{i+\ell n-r}}.$$

Shifting $i \to i-\ell n+r$ in the latter we arrive at (36). $\square$

Remark 6. By (36), we can express Q^r in terms of τ-functions as

$$Q^r = \sum_{\ell=0}^{\infty}\mathrm{diag}\left(\frac{p_\ell(\tilde{D}_{t^{(n)}})\tau^{(n)}_{i-(\ell-1)n+r}\circ\tau^{(n)}_i}{\tau^{(n)}_{i-(\ell-1)n+r}\tau^{(n)}_i}\right)_{i\in\mathbf{Z}} z^{\ell(n-1)}\Lambda^{r-\ell n}. \tag{38}$$

In the case of ordinary discrete KP hierarchy ($n=1$), (38) coincides with the formula (0.13) of the paper [6].

Since $a^{(n,0)}_\ell(i)=0$, then as consequence of the above proposition we deduce following bilinear equations:

$$p_{\ell+1}(\tilde{D}_{t^{(n)}})\tau^{(n)}_{i-\ell n}\circ\tau^{(n)}_i = 0, \quad \ell=0,1,... \tag{39}$$

With the well known bilinear identity for KP wave eigenfunction (see, for example [10])

$$\mathrm{res}_z[(\partial_1^{k_1}...\partial_m^{k_m}\psi(t,z))\cdot\psi^*(t,z)]=0$$

and the fact that $G^{(\ell,\ell n)}_i, \ell=0,1,...$ are purely differential operators, one can deduce the following.

Propositon 9. *τ-functions of nth discrete KP hierarchy satisfy*

$$\mathrm{res}_z[z^\ell\tau^{(n)}_{i+\ell n}(t^{(1)},...,t^{(n)}-[z^{-1}],...)$$

$$\times\tau^{(n)}_i(t^{(1)},...,t^{(n)\prime}+[z^{-1}],...)\exp\xi(t^{(n)}-t^{(n)\prime},z)]=0, \quad \forall t^{(n)},t^{(n)\prime},$$

$$\ell=0,1,2...$$

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